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# SEVERAL GENERALIZATIONS OF TVERBERG'S THEOREM

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ABSTRACT

In a generalization of Radon's theorem, Tverberg showed that each set $S$ of at least $(d+1)(r-1)+1$ points in $R^d$ has an $r$-partition into (pair wise disjoint) subsets $S = S_1 \cup \cdots \cup S_r$ so that $\bigcap_{i=1}^{r} \operatorname{conv} S_i \neq \emptyset$. This note considers the following more general problems: (1) How large must $S \subset R^d$ be to assure that $S$ has an $r$-partition $S = S_1 \cup \cdots \cup S_r$ so that each $n$ members of the family $\{\operatorname{conv} S_i\}_{i=1}^{r}$ have non-empty intersection, where $1 \leqq n \leqq r$. (2) How large must $S \subset R^d$ be to assure that $S$ has an $r$-partition for which $\bigcap_{i=1}^{r} \operatorname{conv} S_i$ is at least 1-dimensional.

## 1. Partial intersections in Tverberg's theorem

Suppose $d > 0$ and $r \geqq n \geqq 1$ are integers. Let $T(d, r, n)$ denote the smallest positive integer with the following property: Every set $S \subset R^d$ of at least $T(d, r, n)$ points has an $r$-partition $S = S_1 \cup \cdots \cup S_r$ into pair wise disjoint sets so that each subfamily of $n$ of the sets in $\{\operatorname{conv} S_i\}_{i=1}^{r}$ has non-empty intersection. A classic theorem of J. Radon asserts that $T(d, 2, 2) = d + 2$ and the generalization of Tverberg [5] takes the form $T(d, r, r) = (d+1)(r-1)+1$. We call the latter number *Tverberg's number* and denote it by $t(d, r)$.

The following lemma collects a number of obvious facts.

LEMMA 1. (a) *$T(d, r, n) \leqq t(d, r)$ for all $d, r, n$, and $=$ holds if $n = r$.*

(b) *$T(d, r, n) = t(d, r)$ if $n \geqq d + 1$.*

*(So assume $1 \leqq n \leqq \min\{d, r\}$ after this.)*

(c) *$T(d, r, 1) = r \leqq t(d, r)$.*

*(So assume $n \geqq 2$ after this.)*

(d) *$T(d, r, n)$ increases monotonically in each variable.*

(e) $T(1, r, n) = \begin{cases} t(1, r) = 2r - 1 & \text{if } n > 1, \\ r & \text{if } n = 1. \end{cases}$

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(*So assume* $d \geqq 2$ *after this.*)

(f) $T(d, r, n) \geqq t(n-1, r)$.

Proof. (a) follows from Tverberg's Theorem, (b) follows from Helly's theorem (see [2]), (c) and (d) are clear from the definitions, (e) follows from (b) and (c), and (f) follows from (b) and (d). □

Lemma 1(c) shows that a weaker intersection condition may lead to a strictly smaller cardinality requirement for $S$ (for the case $n = 1$ at least), since strict inequality holds, i.e. $T(d, r, 1) < t(d, r)$, provided $r \geqq 2$. The following theorem, together with Lemma 1, characterizes $T(d, r, n)$ in the 2-dimensional case, and leads to improved bounds in 3 dimensions.

Theorem 2. *If* $d = n = 2$ (*and* $r \geqq 2$ *arbitrary*), *or if* $d = r = 3$, $n = 2$, *then* $T(d, r, n) = t(d, r)$.

Proof. Lemma 1(a) establishes the inequality one way. For the reverse inequality, it suffices to show that there exists a set $S \subset R^d$ of exactly $(d+1)(r-1)$ points which may not be $r$-partitioned into sets $S = S_1 \cup \cdots \cup S_r$ with $\operatorname{conv} S_i \cap \operatorname{conv} S_j \neq \emptyset$ for all distinct $i, j$. Let $S$ be an $(r-1)$-fold simplicial positive basis for $R^d$, that is, $S = \{\alpha b \mid b \in B, \alpha = 1, 2, \cdots, r-1\}$ where $B$ is a simplicial positive basis for $R^d$, i.e., $B \subset R^d$ has $d+1$ points and the origin lies in the interior of $\operatorname{conv} B$. Suppose, to the contrary, that $S = S_1 \cup \cdots \cup S_r$ and $\operatorname{conv} S_i \cap \operatorname{conv} S_j \neq \emptyset$. Then there exists some $S_i$, say $S_1$, with at most 2 points, by the Pigeon Hole principle. (Remember that $S$ has $(d+1)(r-1)$ points to be partitioned into $r$ disjoint subsets, and $(d+1)(r-1) < 3r$ by the given limitations on $d$ and $r$.) Furthermore, each $S_i$ contains at least 2 points, since if $S_i = \{x\}$, then there exists a hyperplane $H$ through $x$ with at most $r-2$ other points of $S$ on one side of $H$, and $S_i = \operatorname{conv} S_i$ could meet at most $r-2$ other $\operatorname{conv} S_j$. Thus we may suppose that $S_1 = \{\alpha_1 b_1, \alpha_2 b_2\}$ has exactly 2 points of $S$, and by a similar argument with separating hyperplanes, we may assume that $b_1$ and $b_2$ are distinct members of the simplicial positive basis $B$.

It is easy to show that if $\operatorname{conv} S_1 \cap \operatorname{conv} S_i \neq \emptyset$ then $S_i$ must contain at least two points which are multiples of $b_1$ and/or $b_2$. But $S$ contains only $2(r-1)$ multiples of $b_1$ and $b_2$, while each of the $r$ sets $S_i$ must contain two of them. This contradiction establishes the theorem. □

Corollary 2.1. *There exist examples of* $(t(d, r)-1)$*-sets* $S$ *in* $R^d$ *such that any* $r$*-partition* $S = S_1 \cup \cdots \cup S_r$, *for which the sets* $\{\operatorname{conv} S_i\}$ *pair wise intersect, must have* $|S_i| \geqq 3$ *for all* $i$.

COROLLARY 2.2. *Suppose* $r \geqq 3$ *and* $n = 2$ *or* 3 *in the* 3-*dimensional case. Then* $3r = t(3, r) - r + 3 \leqq T(3, r, n) \leqq t(3, r) = 4r - 3$.

PROOF. Lemma 1(a) gives the right inequality and Lemma 1(d) shows that it suffices to assume $n = 2$. To show the left inequality, let $S$ be a partial simplicial basis of cardinality $3r - 1$. (That is, $S$ contains a $[(3r - 1)/4]$-fold simplicial positive basis, and is contained in a $([(3r - 1)/4] + 1)$-fold simplicial positive basis in $R^3$.) Assume, to the contrary, that $S$ has an $r$-partition with the desired properties. As before, some $S_i$ has cardinality 2, and the proof proceeds as in Theorem 2. □

Note that Corollary 2.2 improves the lower bounds given by Lemma 1(f) in the 3-dimensional case.

THEOREM 3. *If* $d = n = r - 1$, *then* $t(d, r) - 1 \leqq T(d, r, n) \leqq t(d, r)$.

PROOF. The right inequality is Lemma 1(a). For the left inequality we show that every set $S$ of $t(d, r) - 2 = ((d + 1)\ (r - 1) + 1) - 2 = rd - 1$ algebraically independent points in $R^d$ fails to have the desired $r$-partition. (Any set of $m$ points in $R^d$ is said to be *algebraically independent* if their $m \cdot d$ real coordinates are algebraically independent over the field of the rationals.) Any $r$-partition of an algebraically independent set $S = S_1 \cup \cdots \cup S_r$ with $rd - 1$ points must have at least one set, say $S_1$, of at most $d - 1$ points. Thus $S_1$ has deficiency at least 2 in $R^d$, i.e., aff $S_1$ (the smallest flat which contains $S_1$) is a translate of a linear subspace of dimension at most $d - 2$. Thus the sum of the deficiencies of some $n = d$ of the sets $S_i$ must be $d + 1$. By the algebraic independence $\bigcap_{i=1}^{n}$ aff $S_i = \emptyset$. But conv $S_i \subseteq$ aff $S_i$, so $S$ can not have the desired $r$-partition. □

Note that if $d = 3$, $n = 3$, $r = 4$, then $12 \leqq T(3, 4, 3) \leqq 13$. (This is a special case of both Theorem 3 and Corollary 2.2.)

CONJECTURE 1. $T(d, r, n) = t(d, r)$ *for all* $r \geqq n \geqq 2$ *and all* $d \geqq 3$.

## 2. Tverberg-type theorems without independence conditions

A set $S \subset R^d$ is said to be $(r, k)$-*divisible* if it can be partitioned into $r$ (pair wise disjoint) subsets whose convex hulls intersect in a set of dimension at least $k$. Thus the theorem of Radon asserts that each set $S \subset R^d$ of at least $d + 2$ points is $(2, 0)$-divisible, while Tverberg's result asserts that each $((d + 1)$ $(r - 1) + 1)$-set $S \subset R^d$ is $(r, 0)$-divisible. Jürgen Eckhoff [3] established the following result while characterizing a certain class of polytopes: Each $(2d + 2)$-set $S \subset R^d$ is $(2, 1)$-divisible. Eckhoff also raised the question, which we now consider, of what the analogous result would be for $(r, 1)$-divisible sets. Similar results have been obtained when the points of $S$ have some sort of independence. Such independence is clearly necessary for $(r, k)$-divisibility if $k \geqq 2$, for otherwise $S$ might lie on a line and no subset could be $k$-dimensional.

THEOREM 4 (Reay [4]). *Each strongly independent* $((d+1)(r-1)+k+1)$-*set* $S \subset R^d$ *is* $(r, k)$-*divisible.*

If either $d = 2$ or $r = 2$, then it may be shown that the strong independence in Theorem 4 may be replaced by the weaker condition that the set is in general position. (See [4].) If we wish to remove all independence conditions (so that $k \leqq 1$), then the following example shows that the sets must be larger than those of Theorem 4.

EXAMPLE. Let $S$ be an $(r-1)$-*fold cross basis* in $R^d$, that is, $S = \{\alpha b \mid b \in B, \alpha = 0, \pm 1, \cdots, \pm(r-1)\}$ where $B$ is any linear basis for $R^d$. Then $S$ is a $(2d(r-1)+1)$-set, and the origin is the only possible *r-divisible point*, i.e., a point $p$ for which there exists an $r$-partition $S = S_1 \cup \cdots \cup S_r$ with $p \in \bigcap_{i=1}^r \operatorname{conv} S_i$. This is easy to see from the fact that each point $p \neq 0$ admits a closed half-space through $p$ which meets $S$ in a $(r-1)$-set, and so for any $r$-partition of $S$, $p \notin \operatorname{conv} S_i$ for some $i$. Hence the set $S$ is not $(r-1)$-divisible. This shows that the bounds in the following are the best possible.

CONJECTURE 2. *Each* $(2d(r-1)+2)$-*set in* $R^d$ *is* $(r, 1)$-*divisible.*

Note that if $r = 2$ this is Eckhoff's result, while if $d = 1$ it reduces to the special case $d = k = 1$ of Theorem 4 (since any set of distinct points in $R^1$ is automatically strongly independent).

THEOREM 5. *Each* $(2d(r-1)+2)$-*set* $S$ *in* $R^d$ *admits two distinct r-divisible points.*

PROOF. Let $f: R^d \to R$ be a continuous linear functional for which $f(x) \neq f(y)$ whenever $x$ and $y$ are distinct points of $S$. The points of $S$ may be labeled in a natural way,

$$S = \{x_1, x_2, \cdots, x_{2d(r-1)+2}\} \qquad \text{so that } f(x_i) < f(x_j) \quad \text{whenever} \quad i < j.$$

Let $S_1 = \{x_i, \cdots, x_{(d+1)(r-1)+1}\}$ and $S_2 = \{x_{(d-1)(r-1)+2}, \cdots, X_{2d(r-1)+2}\}$. Each set $S_i$ contains exactly $(d+1)(r-1)+1$ points of $S$, so Tverberg's theorem implies $z_i \in \bigcap_{k=1}^r \operatorname{conv} S_{ik}$ for some $z_i \in R^d$ and some $r$-partition $S_i = S_{i1} \cup \cdots \cup S_{ik}$. To finish the proof it suffices to show that $z_1$ and $z_2$ are distinct.

Now $f(z_1) \leqq f(x)$ for at least one $x$ in each $S_{1k}$, so $f(z_1) \leqq f(x)$ for at least $r$ of the distinct points of $S_1$. That is, $f(z_1) \leqq f(x_{d(r-1)+1})$. Similar reasoning with $f$ and $S_2$ yields $f(x_{d(r-1)+2}) \leqq f(z_2)$. Thus $f(z_1) < f(z_2)$, so $z_1$ and $z_2$ are distinct $r$-divisible points. □

If the distinct $r$-divisible points of Theorem 5 could use the same $r$-partition of $S$, then $S$ is clearly $(r, 1)$-divisible. Unfortunately there is no guarantee that this is the case. However, the same techniques used in Theorem 5 may be applied to any $(2(d+1)(r-1)+1)$-set $S$ in $R^d$ to get two subsets $S_1$ and $S_2$ with only one point in common, and with an $r$-partition of each $S_i$ for which $z_i \in \bigcap_{k=1}^{r} \operatorname{conv} S_{ik}$ and $z_1 \neq z_2$. Then

$$\{\alpha z_1 + (1-\alpha) z_2 \mid 0 < \alpha < 1\} \subset \bigcap_{k=1}^{r} \operatorname{conv}(S_{1k} \cup S_{2k})$$

so $S$ is $(r, 1)$-divisible. This establishes a crude upper bound on the number of points necessary for $(r, 1)$-divisibility:

Corollary 5.1. *Each* $(2(d+1)(r-1)+1)$*-set* $S$ *in* $R^d$ *is* $(r, 1)$*-divisible.*

The following notation and lemma will lead to a proof of Conjecture 2 for the 2-dimensional case (Corollary 7.1) and to a stronger form of Theorem 5.

For any finite set $S$ in $R^d$, let $D_r(S)$ denote the set of all $r$-divisible points of $S$. Clearly $D_i(S) \supset D_j(S)$ if $i < j$, and $D_1(S) = \operatorname{conv}(S)$. Thus Tverberg's result states that $D_r(S) \neq \varnothing$ if $S$ is a $((d+1)r - d)$-set. Easy examples show that $D_r(S)$ is not a convex set in general. Also let $C_j(S)$ denote the set of all points $y \in R^d$ such that each closed half-space which contains $y$ also contains at least $j$ points of $S$. Equivalently $C_j(S)$ may be defined as the intersection of all closed half-spaces which contain all but $j-1$ or fewer points of $S$, i.e., half-spaces which contain more than $|S| - j$ points of $S$. Clearly $\operatorname{conv} S = C_1(S)$, $C_i(S) \supset C_j(S)$ if $i < j$, and each $C_j(S)$ is a convex polytope. It is well known (see Theorem 2.8 of Danzer–Grunbaum–Klee [2]) that $C_j(S) \neq \varnothing$ if $j \leqq \{|S|/(d+1)\}$. Here, and in the following, $\{m\}$ denotes the smallest integer not less than $m$.

Lemma 6. *Let* $S$ *be any set of* $m$ *points in* $R^2$. *Let* $n = \{m/3\}$, *and* $1 \leqq j < n$. *Then* $C_j(S) = D_j(S)$, *so* $D_j(S)$ *is convex. If* $3 \mid m$ *then this holds for* $j = n$ *as well; in any case* $C_n(S) = \operatorname{conv} D_n(S)$.

Proof. If $p \in R^2 - C_j(S)$ then there exists a closed half-space through $p$ which contains at most $j-1$ points of $S$. Thus $p \notin D_j(S)$ and $C_j(S) \supset D_j(S)$. Birch [1] has shown that the vertices of $C_n(S)$ (and all other points of $C_n(S)$ if $3 \mid m$) are $n$-divisible. This establishes the last half of the lemma. We complete the proof by sketching an argument similar to Birch's to show that each point of

$C_j(S)$ is $j$-divisible.

For any point $q \in C_j(S)$ consider $q$ as the origin in a polar coordinate system for $R^2$, and write the points $x_i = (\rho_i, \theta_i)$ of $S$ in the order of increasing $\theta$. For each $k = 1, \cdots, j$ place $x_i \in S_k$ if $i \equiv k \pmod{j}$. It is easy to show that each $S_k$ has at least 3 points of $S$ and $q \in \operatorname{conv} S_k$. Thus $S = S_1 \cup \cdots \cup S_j$ is a $j$-partition of $S$ with $q \in \bigcap_{i=1}^{j} \operatorname{conv} S_k$ and $q \in D_j(S)$. □

THEOREM 7. *Let $S$ be any $(2d(r-1)+2)$-set in $R^d$. Suppose $D_r(S)$ is convex. Then $S$ is $(r, 1)$-divisible.*

PROOF. By Theorem 5, $D_r(S)$ contains two distinct points and hence a non-degenerate interval $I$. For each $y \in I$ there exists an $r$-partition $S = S_1 \cup \cdots \cup S_r$ with $y \in \bigcap_{i=1}^{r} \operatorname{conv} S_i$. There are only finitely many $r$-partitions of $S$ so some $r$-partition of $S$ has infinitely many points of $I$ in $\bigcap_{i=1}^{r} \operatorname{conv} S_i$. Hence $S$ is $(r, 1)$-divisible. □

The following establishes Conjecture 2 for all values of $r$ when $S$ lies in the plane.

COROLLARY 7.1. *Each $(4(r-1)+2)$-set $S$ in $R^2$ is $(r, 1)$-divisible.*

PROOF. If $r = 2$ this is a special case of Eckhoff's theorem. Suppose $r \geqq 3$. Then $r < \{(4(r-1)+2)/3\}$. Lemma 6 implies $D_r(S)$ is convex. Thus Theorem 7 implies $S$ is $(r, 1)$-divisible. □

COROLLARY 7.2. *With the hypotheses of Theorem 7, the $(2d(r-1)+2)$-set $S$ has a subset $X$ of at most $(d+1)r$ points which is $(r, 1)$-divisible.*

PROOF. Suppose, as in the proof of Theorem 7, there is a non-degenerate interval $I_2 \subset I$ and an $r$-partition $S = S_1 \cup \cdots \cup S_r$ for which $I_2 \subset \bigcap_{i=1}^{r} \operatorname{conv} S_i$. Since each polytope $\operatorname{conv} S_i$ is the (finite) union of simplices with vertices in $S_i$, there exists a non-degenerate interval $I_3 \subset I_2$ and a set $X_i \subset S_i$ of at most $d+1$ points for which $I_3 \subset \bigcap_{i=1}^{r} \operatorname{conv} X_i$. Then $X = \bigcup_{i=1}^{r} X_i$ is an $(r, 1)$-divisible subset of $S$ of at most $(d+1)r$ points. It is interesting to note that either the bound $(d+1)r$ can be reduced or else the $[(d+1)r]$-set $X \subset S$ is actually $(r, d)$-divisible. □

COROLLARY 7.3. *Each $(4(r-1)+2)$-set in $R^2$ contains a subset of at most $3r-1$ points which is $(r, 1)$-divisible.*

PROOF. Corollary 7.2 establishes the existence of such a subset $X$ with at most $3r$ points, i.e., $X$ has an $r$-partition with $\bigcap_{i=1}^{r} \operatorname{conv} X_i$ at least one dimensional. Without loss of generality we may assume interval $I_3$ is an edge of the polygon $\bigcap_{i=1}^{r} \operatorname{conv} X_i$ and thus we may choose some $X_i$ to have only 2 points. □

THEOREM 8. *Each* $(2d(r-1)+2)$*-set* $S$ *in* $R^d$ *has a* $((d+1)(r-1)+2)$*-subset which has two distinct* $r$*-divisible points.*

PROOF. With the notation of Theorem 5 and its proof, we form a sequence $X_1, \cdots, X_t$ of $((d+1)(r-1)+1)$-subsets of $S$. $S_1 = X_1$ and $S_2 = X_t$ and each $X_i$ is obtained from $X_{i-1}$ by replacing one point of $X_i \cap S_1$ by a point from $S_2 - S_1$. (This process changes set $S_1$ into $S_2$ one point at a time.) Each $X_i$ is a $((d+1)(r-1)+1)$-set in $R^d$, so by Tverberg's theorem, there exists a point $w_i \in D_r(X_i)$. Now $w_1 = z_1$ and $w_t = z_2$, so $f(w_1) < f(w_t)$. It follows that for some $i$, $f(w_{i-1}) \neq f(w_i)$. The set $X_{i-1} \cup X_i$ is the desired $((d+1)(r-1)+2)$-subset of $S$ with two distinct $r$-divisible points. □

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